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# Expansion of functions in terms of Chebyshev polynomials

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# EXPANSION OF FUNCTIONS IN TERMS OF CHEBYSHEV POLYNOMIALS

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(PH-31)

Research Department Report No. **1969/10**

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# EXPANSION OF FUNCTIONS IN TERMS OF CHEBYSHEV POLYNOMIALS

SUMMARY

*If $f(x)$ is continuous and of bounded variation in the range $-1 \leqslant x \leqslant +1$, it can be expanded in terms of Chebyshev polynomials. The numerical procedure for such expansion is given in detail. The reverse process of evaluating an expression for which the 'Chebyshev coefficients' are known is also fully considered, for cases in which these coefficients can be more easily derived from the data than a simple, explicit formula for $f(x)$. The process of expansion is closely related to that of Fourier analysis. Such expansion has marked advantages, notably that the error committed by terminating the series at any point can be easily estimated, not only for $f(x)$ but for its integral and derivative. The goodness of fit between this terminated series and $f(x)$ is discussed. Such expansion can be used for the approximate solution of ordinary linear differential equations, particularly those having coefficients which are polynomials of low degree in $x$.*

*This matter is fully discussed by Clenshaw.[1] What follows is essentially a simplified version of Clenshaw's paper, designed to help the engineer who wishes to work to an accuracy of at most three or four significant figures, and to take advantage of the ideas and techniques Clenshaw has so thoroughly established.*

## 1. INTRODUCTION

Suppose we are given a function $f(x)$ which is continuous and bounded in a finite range of $x$, say $a \leqslant x \leqslant b$. If we are able to express $f(x)$ exactly in terms of polynomials, trigonometric, exponential and other elementary functions, this is a great advantage, because any necessary manipulations, including integration and differentiation, are easily performed upon elementary functions. But if for example $f(x)$ is the Bessel function $J_0(x)$ of the first kind of zero order, it is not possible to express it exactly in terms of elementary functions. Approximate expressions for $f(x)$, for a limited range of values of $x$, in terms of elementary functions can be obtained in various ways, but one way which has certain marked advantages is to express $f(x)$ as an infinite series of Chebyshev polynomials, and obtain the required approximation by truncating the series, that is, terminating it after say the $(n+1)$th term. Clenshaw[1] has studied this technique thoroughly, and gives 'Chebyshev coefficients' for various well known mathematical functions to 20 significant figures in most of the cases he studied. He also discusses the relevant properties of Chebyshev polynomials, and means for evaluating approximations directly in terms of the 'Chebyshev coefficients.' Clenshaw's paper includes a bibliography of 28 references, so that it is firmly based on previous work but is also sufficient by itself to enable a mathematically inclined reader to learn the best techniques for expressing a function in terms of Chebyshev polynomials and deriving the maximum advantage with the minimum effort from so doing.

The objective here is much more limited — to enable an engineer to obtain an approximation accurate to at most four or five significant figures for a given function $f(x)$ (which may not be in explicit or closed form, but may instead be say the solution of a differential equation) and to permit easy numerical manipulation of such an approximation when it has been obtained. Clenshaw[1] shows clearly that the 'obvious' ways of carrying out such manipulations are not necessarily the best.

Fundamental properties of Chebyshev polynomials are discussed in Section 2, together with the derivation of 'Chebyshev coefficients' for a given $f(x)$. Fig. 1, which is taken from Reference 2, illustrates the general behaviour of the first six of these polynomials. If an arbitrary function $\phi(x)$ is expressed approximately as a finite series of Chebyshev polynomials, its derivative or integral, and other related functions, can be written down in the same form immediately; the outstanding features of behaviour can then perhaps be inferred qualitatively from Fig. 1 and quantitatively from the procedures discussed below. This can be important where $\phi(x)$ cannot be expressed simply in closed form (for example, because it is only known as the solution of a differential equation or because $\phi(x)$ has a closed form which is complicated or unfamiliar). Economies in the numerical evaluation of $f(x)$ in terms of its 'Chebyshev coefficients' are the subject of Section 3. A particular case is discussed in Section 4. Errors due to truncation of a 'Chebyshev series,' and the goodness of fit between $f(x)$ and the sum of the truncated series, are considered in Section 5. In Section 6 'Chebyshev coefficients' are determined for a function defined as the solution of an ordinary linear differential equation. Special attention is given to the particular case when the differential equation is known to have the expression discussed in Section 4 as an exact, explicit solution. The marked advantages of 'Chebyshev-series' approximations are discussed in Section 7.

## 2. FUNDAMENTAL PROPERTIES OF CHEBYSHEV POLYNOMIALS

We shall follow Clenshaw[1] in taking the definition of the Chebyshev polynomial $T_n(x)$ of order $n$ to be

$$T_n(x) = \cos\,(n\cos^{-1}x)\quad(-1\leqslant x\leqslant 1) \qquad (1)$$

If we put

$$x = \cos\theta \qquad (0\leqslant\theta\leqslant\pi) \qquad (2)$$

then (1) becomes

$$T_n(x) = \cos n\theta \qquad (3)$$

If we require to express $\phi(y)$ in terms of Chebyshev polynomials over the range $a\leqslant y\leqslant b$ instead of $-1\leqslant y\leqslant +1$, it is convenient to write

$$y = \tfrac{1}{2}\left[(b-a)x+(b+a)\right] \qquad (4)$$

and work in terms of $x$ instead of $y$ as much as possible. Likewise, if we require a range $y\geqslant C>0$ for $y$, we should put

$$y = 2C/(x+1) \qquad (5)$$

and, again, work in terms of $x$ as much as possible. We shall therefore assume in what follows that we are required to expand a function $f(x)$ of $x$ approximately

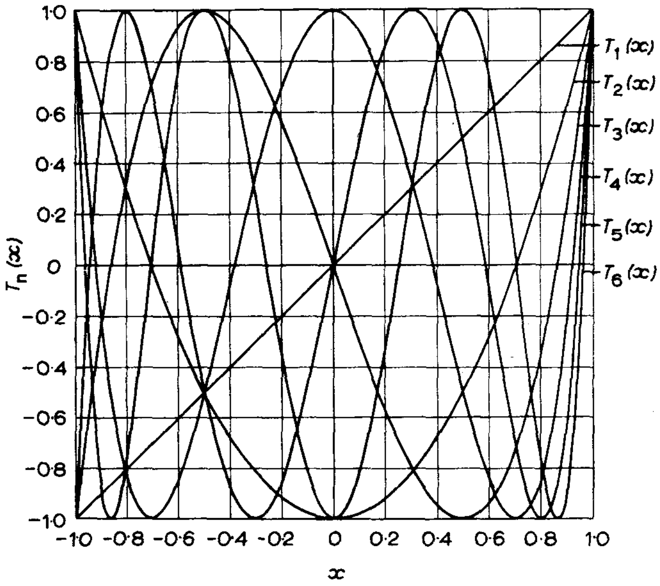


*Fig. 1 - Chebyshev polynomials with the normalisation used in this report*

in terms of Chebyshev polynomials over the range $-1\leqslant x\leqslant 1$. $f(x)$ may be known explicitly, or we may be given information leading to the determination of $f(x)$, such as that $f(x)$ satisfies a differential equation and appropriate boundary conditions.

If we use equations (2) and (3) and work in terms of $\theta$, we see that any relation of the form

$$f(x) = f(\cos\theta) = \tfrac{1}{2}a_0 + a_1\cos\theta + a_2\cos 2\theta + \ldots$$

is in effect expressing as a Fourier series the function $f(\cos\theta)$, which is even in $\theta$ and periodic with period $2\pi$. It therefore follows that

$$a_r = \frac{2}{\pi}\int_0^{\pi} f(\cos\theta)\cos r\theta\, d\theta \qquad (6)$$

and it is also well known that

$$\cos(r+1)\theta + \cos(r-1)\theta = 2\cos\theta\cos r\theta \qquad (7)$$

$$-\int \cos r\theta \sin\theta\, d\theta = \left\{\begin{array}{ll} \cos\theta & r=0 \\ \tfrac{1}{4}\cos 2\theta & r=1 \\ \tfrac{1}{2}\left(\dfrac{\cos[r+1]\theta}{r+1} - \dfrac{\cos[r-1]\theta}{r-1}\right) & r>1 \end{array}\right\} \qquad (8)$$

$$\int_0^{\pi} \cos r\theta\cos s\theta\, d\theta = \left\{\begin{array}{l} \pi\ (r=s=0) \\ \pi/2\ (r=s\neq 0) \\ 0\ (r\neq s) \end{array}\right\} \qquad (9)$$

By substituting from (2) and (3), we can express equations (6), (7), (8) and (9) directly in terms of $x$ in the form

$$a_r = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)\, T_r(x)\, dx}{(1-x^2)^{1/2}} \tag{10}$$

$$T_{r+1}(x) - 2x\, T_r(x) + T_{r-1}(x) = 0 \tag{11}$$

$$\int T_r(x)\, dx = \begin{cases} T_1(x) & r = 0 \\ \tfrac{1}{4} T_2(x) & r = 1 \\ \tfrac{1}{2}\left[\dfrac{T_{r+1}(x)}{r+1} - \dfrac{T_{r-1}(x)}{r-1}\right] & r > 1 \end{cases} \tag{12}$$

$$\int_{-1}^{1} T_r(x)\, T_s(x)\, (1-x^2)^{-1/2} dx = \begin{cases} \pi \ (r = s = 0) \\ \tfrac{1}{2}\pi \ (r = s \neq 0) \\ 0 \ (r \neq s) \end{cases} \tag{13}$$

Although (10), which is equivalent to (6), appears to be the obvious way of deriving the coefficients $a_r$ when $f(x)$ is known explicitly, it is not necessarily the best way. It can be shown that if $n>0$ and $r$ and $s$ are both $\leqslant n$

$$\tfrac{1}{2}\left[1 + (-1)^{r+s}\right] + \sum_{q=1}^{n-1} \cos\frac{r\pi q}{n} \cos\frac{s\pi q}{n} = \begin{cases} n \ (r = s = 0 \text{ or } n) \\ \tfrac{1}{2} n \ (r = s \neq 0 \text{ or } n) \\ 0 \ (r \neq s) \end{cases} \tag{14}$$

and this, expressed in terms of $x$, is

$$\tfrac{1}{2}\left[1 + (-1)^{r+s}\right] + \sum_{q=1}^{n-1} T_r(x_q)\, T_s(x_q) = \begin{cases} n \ (r = s = 0 \text{ or } n) \\ \tfrac{1}{2} n \ (r = s \neq 0 \text{ or } n) \\ 0 \ (r \neq s) \end{cases} \tag{15}$$

where

$$x_q = \cos(q\pi/n) \tag{16}$$

If we define

$$\alpha_r = \frac{2}{n}\left[\tfrac{1}{2} f(1) + \tfrac{1}{2}(-1)^r f(-1) + \sum_{q=1}^{n-1} f(x_q)\, T_r(x_q)\right] \tag{17}$$

then it can be shown by means of (15) that if $f(x)$ is a polynomial of degree less than $n$

$$\alpha_r = a_r \tag{18}$$

whereas in general

$$\alpha_r = a_r + a_{2n-r} + a_{2n+r} + a_{4n-r} + a_{4n+r} + \dots \tag{19}$$

But if $n$ is sufficiently large, we can neglect $(\alpha_r - a_r)$ in (19), and write

$$f(x) \approx \tfrac{1}{2}\alpha_0 + \sum_{r=1}^{n} \alpha_r\, T_r(x) \tag{20}$$

where $\alpha_r$ is given by (17). We note that (17) involves no integration, but only the values of $f(x)$ at $x=1$, $x=-1$ and $(n-1)$ other values of $x$ which depend only upon $n$ and not upon the nature of $f(x)$. Usually (17) is a better formula for determining the Chebyshev coefficients than (10), and an Elliott-803 computer programme based on (17) has been fully tested.

## 3. EVALUATION OF A FUNCTION IN TERMS OF ITS CHEBYSHEV COEFFICIENTS

We now assume that we have a function $f(x)$ such that

$$f(x) \approx \tfrac{1}{2}a_0 + \sum_{r=1}^{n} a_r\, T_r(x) \tag{21}$$

where the $a_r$ have been determined previously, as in Section 2 or otherwise, and $n$ has been chosen sufficiently large for the 'truncation error' introduced by terminating the right-hand side of (21) at the $(n+1)$th term to be negligible. (This and related errors are discussed in Section 5.) If we require $f(x)$ for a given $x$, a possible approach is to substitute from (1), so that $f(x)$ is approximated by means of a polynomial of degree $n$. But if we decide that the accuracy is not good enough, so that the value of $n$ must be increased, the $a_r$ already calculated remain the same, whereas the coefficients of a particular power of $x$ below the $n$th do not. Consider therefore the alternative possibility of evaluating successively the quantities $b_n, b_{n-1}, b_{n-2} \dots b_2, b_1, b_0$ where

$$b_r = 2x\, b_{r+1} - b_{r+2} + a_r \quad (b_{n+1} = b_{n+2} = 0) \tag{22}$$

Then it can be shown from (11) that

$$f(x) = \tfrac{1}{2}(b_0 - b_2) \tag{23}$$

Clenshaw[1] gives modified forms of (22) when it is known that alternate coefficients $b_r$ will be zero.

This modification is based upon the fact that

$$T_{2r}(x) = T_r(2x^2 - 1) \tag{24}$$

but for our present purpose it is probably simpler to apply directly (22) in all cases for moderate $n$.

## 4. A PARTICULAR CASE

Suppose that we wish to find the 'Chebyshev series' appropriate to the expression

$$y_1 = e^{2x} + 2x + 5 \qquad (-1 \leqslant x \leqslant +1) \tag{25}$$

and that we require the result to be accurate to three decimal places, so that it will be assumed that calculations based upon five-figure tables are sufficiently accurate. We shall also assume that 8 is a sufficiently high value of $n$. (If at a later stage it seems necessary to increase $n$, it will be best to *double* $n$, so that all the calculation already done is still useful.) The 'Chebyshev coefficients' $\alpha_r$ are given in Table 1.

## 5. GOODNESS OF FIT AND TRUNCATION ERRORS

The agreement between $y_1$ given by (25) and the first nine terms of the corresponding Chebyshev series has clearly been indicated in Table 2; it is appropriate now to consider more generally the errors associated with the truncation of Chebyshev series.

Suppose that we seek to approximate to $f(x)$ by means of an expression

$$y = \tfrac{1}{2}a_0 + a_1 T_1(x) + a_2 T_2(x) + \dots + a_r T_r(x) \quad (r<n) \qquad (26)$$

and that we do this by trying to satisfy the equations

$$f(x_q) = \tfrac{1}{2}a_0 + a_1 T_1(x_q) + a_2 T_2(x_q) + \dots + a_r T_r(x_q) \qquad (27)$$

where

$$x_q = \cos(q\pi/n),\ q = 0,\ 1,\ 2 \dots n \qquad (28)$$

Since there are only $(r+1)$ unknown coefficients $a_0, a_1, a_2 \dots a_r$ and $(n+1)$ equations (27), we cannot in general succeed; our best hope is to attempt a solution by least squares, which means that we multiply equation (27) by $\tfrac{1}{2}T_s(1)$ or $\tfrac{1}{2}$ when $q=0$, by $\tfrac{1}{2}T_s(-1)$ or $\tfrac{1}{2}(-1)^s$ when $q=n$, and by $T_s(x_q)$ for all other values of $q$, and add. Because of the relations (15), the 'least-squares' equations determining the best available values of the coefficients $a_r$ take the remarkably simple form

$$a_s = \frac{2}{n}\left[\tfrac{1}{2}f(1) + \tfrac{1}{2}(-1)^n f(-1) + \sum_{q=1}^{n-1} f(x_q)\,T_s(x_q)\right] \qquad (29)$$

Thus the polynomial (26) can be described as the least-squares best-fitting polynomial of degree $r$ to $f(x)$, when the fitting is carried out in the manner indicated by equation (27). If it is decided that the fit given by (26) is not good enough, and that the value of $r$ must be increased, the coefficients of (26) already calculated will not be altered when $r$ is increased. (If $x$ is replaced by $\cos\theta$, these results have well known counterparts in Fourier-analysis.)

Clenshaw[1] points out that if $f(x)$ is continuous and of bounded variation in the interval $-1 \leqslant x \leqslant +1$, there is an expansion of the form

$$f(x) = \tfrac{1}{2}a_0 + a_1 T_1(x) + \dots \qquad (30)$$

which is uniformly convergent throughout the range. It is also clear that none of the $T_s(x)$ can be numerically greater than 1 in the range, and therefore we can say immediately that the error due to termination of the Chebyshev series (30) at the $(n+1)$th term must be numerically less than

$$\sum_{r=n+1}^{\infty} |a_r| \qquad (31)$$

and the above-mentioned uniform convergence guarantees the arbitrary smallness of (31) for sufficiently large $n$.

*TABLE 1*

*Chebyshev Coefficients $\alpha_r$ for y, Given by Equation (25)*

| $r$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $\alpha_r$ | 14·55923 | 5·18133 | 1·37794 | 0·42548 | 0·10147 | 0·01966 | 0·00317 | 0·00039 | 0·00003 |

*TABLE 2*

*Comparison of $y_1$ and the Sum $Y_1$ of the First Nine Terms of its 'Chebyshev Series'*

| $x$ | −1 | −0·75 | −0·5 | −0·25 | 0 | 0·25 | 0·5 | 0·75 | 1 |
|---|---|---|---|---|---|---|---|---|---|
| $y_1$ | 3·13534 | 3·72313 | 4·36788 | 5·10653 | 6·0 | 7·14872 | 8·71828 | 10·98169 | 14·38906 |
| $Y_1$ | 3·13537 | 3·72314 | 4·36786 | 5·10646 | 6·00001 | 7·14882 | 8·71828 | 10·98172 | 14·38909 |

By repeated use of (12), Clenshaw[1] shows that if $f(x)$ has a Chebyshev-series expansion given by (30), then

$$\int f(x)\mathrm{d}x = \text{const.} + \tfrac{1}{2}a_0 T_1(x) + \tfrac{1}{4}a_1 T_2(x) + \sum_{r=2}^{\infty} \frac{a_r}{2}\left(\frac{T_{r+1}(x)}{r+1} - \frac{T_{r-1}(x)}{r-1}\right)$$

$$= \text{const.} + \sum_{r=1}^{\infty} A_r T_r(x) \qquad (32)$$

where

$$A_r = \frac{a_{r-1} - a_{r+1}}{2r} \qquad (33)$$

Hence given the $a_r$, we can deduce the $A_r$ directly from (33), and the $A_r$ will become negligible for a lower value of $r$ than the $a_r$. If on the other hand we want to differentiate a Chebyshev series, so that we are in effect given the $A_r$ and require the corresponding $a_r$, we must first determine $n$ so that $rA_r$ is negligible for $r \geqslant n$, and neglect $a_r$ for $r \geqslant n$; repeated use of (33) then gives $a_{n-1}$, $a_{n-2}$ etc in succession. The above remarks mean that once a Chebyshev series has been determined for $f(x)$, we can derive the Chebyshev series also for the integral and derivative of $f(x)$, and adapt (31) to estimate the 'truncation errors' of these derived series.

## 6. CHEBYSHEV APPROXIMATION FOR THE SOLUTION OF AN ORDINARY LINEAR DIFFERENTIAL EQUATION

Hitherto we have considered the determination of the 'Chebyshev coefficients' when $f(x)$ is given explicitly, and the reverse process of evaluating $f(x)$ directly if we happen to know its Chebyshev coefficients. In this section we consider the case when $f(x)$ is not initially known explicitly, but is the solution of an ordinary linear differential equation with appropriate boundary conditions. This case has been left until the end because the arithmetical and algebraic details are somewhat more complicated; no difference in principle is involved.

Suppose that we define $f(x)$ as the solution of the ordinary linear differential equation

$$(x+2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - (2x+5)\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0 \qquad (34)$$

with the boundary conditions

$$y = 6 \text{ and } \frac{\mathrm{d}y}{\mathrm{d}x} = 4 \text{ when } x = 0 \qquad (35)$$

Piaggio[3] studies the differential equation (34) in considerable detail; the solution satisfying the boundary conditions (35) is $y = y_1$, where $y_1$ is given by (25). If therefore we assume that an approximate solution of (34) for the range $-1 \leqslant x \leqslant +1$ is such that

$$\left.\begin{aligned} y = \tfrac{1}{2}a_0 + \sum_{r=1}^{8} a_r T_r(x): \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= \tfrac{1}{2}a_0' + \sum_{r=1}^{8} a_r' T_r(x); \\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \tfrac{1}{2}a_0'' + \sum_{r=1}^{8} a_r'' T_r(x) \end{aligned}\right\} \qquad (36)$$

we seek to determine the coefficients $a_r$ directly from (34) and (35); any result obtained must be consistent with values obtained from Table 1.

Now from (12), we have for $r = 1, 2 \ldots. 8$

$$a_{r-1}'' = a_{r+1}'' + 2ra_r' \qquad (37)$$

$$a_{r-1}' = a_{r+1}' + 2ra_r \qquad (38)$$

and we can obtain a set of linear recurrence relations between various $a_r$, $a_r'$ and $a_r''$ from the differential equation (34) itself if we first note that

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \approx x\left[\tfrac{1}{2}a_0'' + \sum_{r=1}^{8} a_r'' T_r(x)\right]$$

$$= \tfrac{1}{2}\left[a_1'' T_0(x) + \sum_{r=1}^{9} (a_{r-1}'' + a_{r+1}'') T_r(x)\right] \qquad (39)$$

it being understood that $a_{r+1}''$ is replaced by 0 if $r > 7$.

Similarly

$$x\frac{\mathrm{d}y}{\mathrm{d}x} \approx \tfrac{1}{2}\left[a_1' T_0(x) + \sum_{r=1}^{9} (a_{r-1}' + a_{r+1}') T_r(x)\right] \qquad (40)$$

where $a_{r+1}'$ is also taken as zero for $r > 7$.

Substituting into (34), we thus obtain

$$2a_0 - 5a_0' + 2a_0'' - 2a_1' + a_1'' = 0 \qquad (41)$$

$$-a_{r-1}' + \tfrac{1}{2}a_{r-1}'' + 2a_r - 5a_r' + 2a_r'' - a_{r+1}' + \tfrac{1}{2}a_{r+1}'' = 0$$

$$(r = 1,\ 2 \ldots.\ 8) \qquad (42)$$

Equation (41) expresses the fact that the coefficient of $T_0(x)$ in (34) is zero, while equation (42) expresses the fact that the coefficients of $T_1(x)$, $T_2(x)$, .... $T_8(x)$ are zero.

Equations (37), (38), (41) and (42) give us 25 linear relations between the 27 quantities $a_r$, $a_r'$ and $a_r''$ and therefore in general we should be able to express all these quantities as linear combinations of any two of them. Let us therefore attempt to express all these quantities in terms of $a_8$ and $a_8'$, and assume that $a_r = a_r' = a_r'' = 0$ if $r > 8$. The easiest way to do this is to start by taking $a_8$ arbitrary, $a_8' = 0$, then repeat the process taking $a_8 = 0$, $a_8'$ arbitrary, and add the results. In this way we should usually be able to express all the $a_r$ etc as linear combinations of $a_8$ and $a_8'$; the results could then be substituted in the boundary conditions (35) to determine the appropriate values of $a_8$ and $a_8'$.

If we know $a_8$ and $a_8'$, (37) with $r = 8$ gives $a_7''$ and (38) with $r = 8$ gives $a_7'$; thereafter (42) with $r = 8$ gives $a_8''$.

When we come to $r = 7$, (37) determines $a_6'$ immediately but (38) and (42) have both $a_7$ and $a_6'$ as unknowns, so it is desirable to eliminate $a_{r-1}'$ from (42) by means of (38), and rewrite it in the form

$$(2r - 2)a_r = \tfrac{1}{2}a_{r-1}'' - 5a_r' + 2a_r'' - 2a_{r+1}' + \tfrac{1}{2}a_{r+1}'' \qquad (43)$$

Putting $r = 7$ in (43) determines $a_7$ and then putting $r = 7$ in (38) determines $a_6'$. Then we reduce $r$ to 6, substitute into (37) to obtain $a_5''$, into (43) to obtain $a_6$ and into (38) to obtain $a_5'$, and so on; the process is straightforward until we come to the last stage when $r = 1$ and we have determined all the coefficients except $a_1$, $a_0$, $a_0'$ and $a_0''$.

When $r = 1$, (37) can be used as usual to obtain $a_0''$ but (43) does not give any information about $a_1$. Instead, equating the right-hand side of (43) to zero determines the ratio of $a_8'$ to $a_8$, and thus permits us to express all the coefficients hitherto obtained as multiples of $a_8$. The ratio $a_8'/a_8$ is 2·000, and in Table 3 the coefficients $a_r$ (except $a_1$), $a_r'$ and $a_r''$ are expressed in the form

$$a_r = \lambda_r a_8 + \mu_r a_8' = \nu_r a_8; \quad a_r' = \lambda_r' a_8 + \mu_r' a_8' = \nu_r' a_8$$
$$a_r'' = \lambda_r'' a_8 + \mu_r'' a_8 = \nu_r'' a_8 \qquad (44)$$

for $r \geqslant 1$; the coefficients $a_0$, $a_0'$ and $a_0''$ are expressed as linear combinations of $a_8$ and $a_1$ by means of (37), (38), and (41).

Substituting into the boundary conditions (35) we therefore find

$$a_8 = 5{\cdot}61459 \times 10^{-5} \qquad a_0 = 14{\cdot}55924 \qquad (45)$$

and thus the final values of $a_0$, $a_1$ .... $a_8$ are those given in Table 4; they are in very good agreement with the values of the quantities $\alpha_r$ in Table 1 as they should be.

## 7. ADVANTAGES OF CHEBYSHEV-SERIES APPROXIMATIONS

In the foregoing, we have shown, following Clenshaw,[1] how to derive the coefficients of a Chebyshev-series approximation preferably by means of equation (17) and how to evaluate economically $f(x)$ given its Chebyshev coefficients preferably by means of equation (23). We have also considered the case in which $f(x)$ is given not explicitly but as the solution of a linear differential equation (Section 6).

*TABLE 3*

| $r$ | $\lambda_r$ | $\mu_r$ | $\nu_r$ | $\lambda_r'$ | $\mu_r'$ | $\nu_r'$ | $\lambda_r''$ | $\mu_r''$ | $\nu_r''$ |
|---|---|---|---|---|---|---|---|---|---|
| 8 | 1 | 0 | 1 | 0 | 1 | 2 | 7 | −1·5 | 4 |
| 7 | 3·25 | 2·375 | 8 | 16 | 0 | 16 | 0 | 16 | 32 |
| 6 | 47·55 | 4·725 | 57 | 45·5 | 34·25 | 114 | 231 | −1·5 | 228 |
| 5 | 154 | 98 | 350 | 586·6 | 56·7 | 700 | 546 | 427 | 1400 |
| 4 | 1663·55 | 71·725 | 1806·05 | 1585·5 | 1014·25 | 3614 | 6097 | 565·5 | 7228 |
| 3 | 399 | 3589·5 | 7578 | 13895 | 630·5 | 15156 | 13230 | 8541 | 30312 |
| 2 | 76217·75 | −25838·375 | 24541 | 3979·5 | 22551·25 | 49082 | 89467 | 4348·5 | 98164 |
| 1 | – | – | – | 318766 | −102723 | 113320 | 29148 | 98746 | 226640 |

*TABLE 4*

*Final Values of $a_r$*

| $r$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $a_r$ | 14·55924 | 5·18126 | 1·37788 | 0·42547 | 0·10140 | 0·01965 | 0·00320 | 0·00045 | 0·00006 |

Note that except for $r = 0$, 1, $4a_r = 2a_r' = a_r''$; for $r = 0$, 1 the values of $a_r'$, $a_r''$ are $a_0' = 13{\cdot}11826$; $a_0'' = 12{\cdot}23647$; $a_1' = 6{\cdot}36245$; $a_1'' = 12{\cdot}72491$.

Now the outstanding advantage of Chebyshev-series expansion is that coefficients can be determined explicitly from the straightforward formula (29) whatever the degree of the approximating formula, and that if it is later necessary to determine an approximating formula of higher degree, the coefficients already determined do not change (as they would if we sought directly to approximate by means of a curve of degree $r$). Then there is seldom any difficulty about the convergence of a Chebyshev series; the function $f(x)$ represented over the range $-1 \leqslant x \leqslant +1$ has only to be such that $f(\cos\theta)$ has a Fourier series in $\theta$. Again, the expression (31) for the 'truncation error' is remarkably simple, and so is the formula (33) by means of which Chebyshev series for the integral and derivative of $f(x)$ can be derived at the same time as that for $f(x)$ itself.

## 8. REFERENCES


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